Chapter 15 – Solutions of Triangles

In general we will be given at least two of the three sides and an angles of a triangle, and asked to find one or more of the remaining sides of the angles.

We label the vertices of the triangle A, B and C and the sides opposite each angle are then labeled a, b, and c respectively. When we have a right angle triangle (as in section 15.1, 15.2 and 15.3) we will assume the right angle is at C and thus c is the hypotenuse.

The following methods are for the trigonometrical scales on the slide, interchanging the C scale for D, and CI scale for DI, and vise versa, will produce a suitable method.

15.1 Right Angle Triangle (given 2 sides)
We know angle C = 90° and sides a and b. Angle A and B, and the hypotenuse c are unknown. Thus we use

$$\tan A = \frac{a}{b}$$

B = 90° - A

And $\quad c = \dfrac{a}{\sin A}$ (from $\sin A = \dfrac{a}{c}$)

Example: Given a = 44.5 cm and b = 59 cm
To find angles A, B and the hypotenuse c.
1. Set the hair line over 44.5 on the D scale.
2. Place the left index of the CI scale under the hair line. (In some examples it will be the right index).
3. Reset the hair line read off 37° on the T scale as the value for A.
4. Reset the hair line over 37° on the S scale.
5. Under the hair line read off 74 on the CI scale as the value for c.
   Then B = 90° - A = 90° - 37° = 53°
   Thus A = 37°, B = 53° and c = 74cm

Exercise 15(a)
(Note C = 90° and side c is the hypotenuse.)
Find angles A and B, and side c, given:
   (i)   a = 14.1 cm and b = 9.6 cm
   (ii)   a = 150 cm and b = 83 cm
   (iii)   a = 3.6 cm and b = 5.9 cm
   (iv)   a = 13 cm and b = 30.4 cm

15.2 Right Angle Triangle (given the hypotenuse and an angle).
We know angles C = 90° and A, and side c. Then sides a and b and the angle B are unknown.

Thus we use a = c sin A (from $\sin A = \dfrac{a}{c}$)

b = c cos A (from $\cos A = \dfrac{b}{c}$)

and   B = 90° - A

Example; Given c = 9.5 cm and B = 41°
To find angle A and sides a and b.
1. Place 9.5 of the C scale over the right index of the D scale.
2. Set the hair line over 41° on the S scale.
3. Under the hair line read off 6.23 on the C scale as the value for a.
4. Reset the hair line over 41° (red) on the S scale.

5. Under the hair line read off 7.16 on the C scale as the value for b.
    B = 90° - A = 90° - 41° = 49°
    Thus B = 49°, a = 6.23 cm and b – 71.6 cm

Exercise 15(b)
(Note C = 90° and side c is the hypotenuse.)
Find angles B and sides a and b, given:

    (i)   A = 34°18' and c = 18.5cm
    (ii)   A = 48.4° and c = 42.5cm
    (iii)   A = 26° and c = 7.3cm
    (iv)   A = 56.2° and c = 315cm

15.3 Right Angle Triangle (given the hypotenuse and a side)
We know angle C = 90° and sides c and a. The angles A and B and side b are unknown.

Thus we use

$$\sin A = \frac{a}{c}$$
$$B = 90 - A$$

and $\qquad\qquad b = c \sin B \;\; (\text{from } \sin B = \frac{b}{c})$

Example: Given a = 31.5 cm and c = 79
To find angles A and B and side b.

1. Place the 79 of the C scale over the right index of the D scale.
2. Set the hair line over 31.5 on the C scale.
3. Under the hair line read off 23.5° on the S scale as the value for A.
    Then B = 90° - A = 90° - 23.5° = 66.5°
4. Reset the hair line over 66.5° on the S scale.
5. Under the hair line read off 72.4 on the C scale as the value for b.
    Thus A = 23.5°, B = 66.5° and b = 74.2 cm

Exercise 15(c)
(Note Angle C = 90° and side c is the hypotenuse.)
Find angles A and B and side b given:

    (i)   a = 43 cm and c = 74 cm
    (ii)   a = 17.2 cm and c = 29.6 cm
    (iii)   a = 35 cm and c = 52 cm
    (iv)   a = 6.3 cm and c = 15.8 cm

15.4 Area of a Triangle
Knowing two sides and the included angle of any triangle, its area is given by:

        Area = ½ a b sin C,
        Or   = ½ b c sin A,
        Or   = ½ a c sin B.

Example: Given a =15 cm., b = 17 cm and C = 36°.
        ∴ area = ½ x 15 x 17 x sin 36°

1. Set the hair line over 36° on the S scale.
2. Place the 15 of the CI scale under the hair line.
3. Reset the hair line over 17 on the C scale.
4. Place the 2 of the C scale under the hair line.
5. Below the right index of the C scale read off 75 on the D scale as the answer.
    i.e. area = 75 cm^2

Exercise 15(d)
Find the area of the following triangles given that:
  (i)   a = 6 cm, b = 9 cm and C = 38°.
  (ii)   b = 4.1 cm, c = 2.9 cm and A = 67°.
  (iii)   a = 15.4 cm, c = 13.8 cm and B = 43°24'.
  (iv)   a = 29.1 cm, b = 35.3 cm and C 105°.

15.5 Sine Rule (Scalene Triangles)
The sine rule is usually expressed as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

In this unit we will deal with the cases where two angles and a side, or two sides and an angle opposite one of the sides are given. See the Appendix for cases where two sides and an angle not opposite one of the sides, or three sides are given.

Example 1: Given a 48 = cm, A = 55° and B = 76°
To find an angle C and sides b and c.
          C = 180° - (55° + 76°)
          ∴ C = 49°

Thus we have: $\dfrac{\sin 55°}{48} = \dfrac{\sin 76°}{b} = \dfrac{\sin 49°}{c}$
  1. Set the hair line over 55° on the scale.
  2. Place the 48 of the C scale under the hair line. (This sets up the known ratio.)
  3. Reset the hair line over 76° on the S scale.
  4. Under the hair line read off 56.8 on the C scale as the value for b.
  5. Reset the hair line over 49° on the S scale.
  6. Under the hair line read off 44.2 on the C scale as the value for c.
      ∴ C = 49°, b = 56.8 cm and c = 44.2 cm

Note:
  (a) In the above procedure we have set up the ratio of $\dfrac{\sin 55°}{48}$ on the C and S scales. Thus for a given angle on the S scale the side opposite it is read directly off the C scale, and for a given side on the C scale the angle opposite it is found on the S scale.
  (b) Instead of using the C scales, we could have used the CF and S scales. (See example 2 below).
  (c) For an S' scale on the slide (or a Slide Rule with the S scale on the slide), it can be used with the D scale on the body of the Slide Rule.)

Example 2: Given a = 8.3 cm, A = 48.8° and B = 60°
To find angle C and sides b and c.
          C = 180° - (48.4° + 60°)
          ∴ C = 71.6°

Thus we have: $\dfrac{\sin 48.4°}{8.3} = \dfrac{\sin 60°}{b} = \dfrac{\sin 71.6°}{c}$

  1. Set the hair line over 48.4° on the scale.
  2. Place the 8.3 of the CF scale under the hair line.
  3. Reset the hair line over 60° on the S scale.
  4. Under the hair line read off 9.6 on the CF scale as the value for b.
  5. Reset the hair line over 71.6° on the S scale.

6.   Under the hair line read off 10.51 on the CF scale as the value for c.
     $\therefore$ C = 71.6°, b = 9.6 cm and c = 10.51 cm

Note: Had we set up Example 2 on the C and S scales, we would have found that for the hair line over 71.6° on the S scale, it would have been off the end of the C scale. In such circumstances a Slide Rule such as the Faber-Castle 2/83N or 2/82N with extended scale graduations can be a great help.

CASE II – Two Sides and an Angle Opposite One of Them (S.S.A)
   A.   One Triangle Possible, as the side opposite the given angle is longer than the other side given.

Example 3: Given A = 59°, a = 7.8 cm and c = 6.2 cm
To find angles B and C and side b.

Thus we have: $\dfrac{\sin 59°}{7.8} = \dfrac{\sin \mathbf{B}}{b} = \dfrac{\sin C}{6.2}$

   a)   Set the hair line over 59° on the S scale.
   b)   Place the 7.8 of the C scale under the hair line.
   c)   Reset the hair line over 6.2 on the C scale.
   d)   Under the hair line read off 43° on the S scale as the value for angle C.
        Thus B = 180° - (59° + 43°)
        $\therefore$ B = 78°
   e)   Reset the hair line over 78° on the S scale.
   f)   Under the hair line read off 8.9 on the C scale as the value for b.

   B.   Two Triangles Possible, as the side opposite the given angle is shorter than the other side give.

Example 4: Given a = 17.2 cm, b = 19.6 cm and A = 51°.
To find angles B and C, and side c.

Thus we have: $\dfrac{\sin 51°}{17.2} = \dfrac{\sin \mathbf{B}}{19.6} = \dfrac{\sin C}{c}$

   1.   Set the hair line over 51° on the S scale.
   2.   Place the 17.2 of the CF scale under the hair line.
   3.   Reset the hair line over 19.6 on the CF scale.
   4.   Under the hair line read off 62.3° on the S scale as one of the possible values for angle B.
        $\therefore$ B_1 = 62.3° Then B_2 = 180 – 62.3°
                              = 117.7°
        Thus C_1 = 180° - (51° + 62.3°)
                = 66.7°
        and C_2 = 180° - (51° + 117.7°)
                = 11.3°
For C_1 = 66.7°
   5.   Reset the hair line over 66.7° on the S scale.
   6.   Under the hair line read off 21.5 on the CF scale as the value for c_1
For C_1 = 117.7°
   7.   Reset the hair line over 11.3° on the S scale.
   8.   Under the hair line read off 4.34 on the CF scale as the value for c_2
Thus the two possible answers are:
B_1 = 62.3°, C_1 = 66.7° and c_1 = 21.5 cm
B_2 = 117.7°, C_2 = 11.3° and c_2 = 4.34 cm
(Note: we used the CF scale with the S scale as it would otherwise have run off the end of the C scale.)

Exercise 15(e)
Use the Sine Rule to find the remaining sides and angles, given:
   (i)   A = 66°, C = 48° and a = 42cm

(ii)   B = 75°, C = 62° and a = 10.4cm
(iii)   A = 126°, B = 15° and c = 17cm
(iv)   a = 73cm, c = 28cm and A = 42°
(v)   a = 9.1cm, b = 7.7cm and A = 51.2°
(vi)   b = 25.4cm, c = 3.3cm and B = 100°
(vii)   b = 83cm, c = 96.8cm and B = 48.4°
(viii) a = 62.5cm, b = 98cm and A = 38.3°

15.6 Cosine Rule (Scalene Triangles)

The cosine rule is usually expressed as:

$$a^2 = b^2 + c^2 - 2bc\cos A \ ,$$

or $$b^2 = a^2 + c^2 - 2ac\cos B \ ,$$

or $$c^2 = a^2 + b^2 - 2ab\cos C \ .$$

These also can be expressed in the form – $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, etc.

We use the slide rule to square, multiply, divide and obtain square roots in these expressions.

Exercise 15(f)

Use the cosine and sine rules to solve the triangles, given:

(i)   a = 45cm, b = 20cm and c = 60cm
(ii)   a = 2.2cm, b = 3.4cm and c = 2.4cm
(iii)   a = 25cm, b = 35cm and C = 38°
(iv)   b = 14.9cm, c = 7.24cm and A = 104°36'